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Convergence analysis of a family of 14-node brick elements *

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Abstract

In this paper, we will give convergence analysis for a family of 14-node elements which was proposed by I. M. Smith and D. J. Kidger in 1992. The 14 DOFs are taken as the value at the eight vertices and six face-centroids. For second-order elliptic problem, we will show that among all the Smith-Kidger 14-node elements, Type 1, Type 2 and type 5 elements can get the optimal convergence order and Type 6 get lower convergence order. Motivated by the proof, we also present a new 14-node nonconforming element. If we change the DOFs into the value at the eight vertices and the integration value of six faces, we show that Type 1, Type 2 and Type 5 keep the optimal convergence order and Type 6 element improve one order accuracy which means that it also get optimal convergence order.

Keywords: Nonconforming element; Brick element; 14-node element; Second-order elliptic problem; Smith-Kidger element

1 Introduction

Until now, there are many three-dimensional brick elements. For conforming case, the trilinear element, 27-node element and Seredipity elements are well known. For nonconforming case, Rannacher-Turek [7] presented the rotated Q_1 nonconforming element, Douglas-Santos-Sheen-Ye [2] introduced the nonconforming finite element using

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only six values at the centroids of the faces as degrees of freedom, and Park-Sheen presented a P_1-nonconforming finite element on cube meshes which has the lowest degrees of freedom [6]. Wilson also defined a linear-order nonconforming brick element [1, 15] with 11 DOFs whose basis consists of trilinear polynomials plus $\{1-x^2, 1-y^2, 1-z^2\}$ on $\widehat{\mathbf{K}} = [-1,1]^3$. This element obtains an $O(h)$ convergence rate in energy norm. (see [1, Page 217, Remark 4.2.3])

To get a higher order of convergence for error estimates, Smith and Kidger [12] successfully developed three-dimensional brick elements of 14 DOFs. They investigated six most possible 14 DOFs elements systematically considering the Pascal pyramid, and concluded that the type 1 (as well as type 2) and type 6 elements are successful ones. The type 1 element add the span of four nonsymmetric cubic polynomials $\{xyz, x^2y, y^2z, z^2x\}$ while the type 6 element the span of $\{xyz, xy^2z^2, x^2yz^2, x^2y^2z\}$ to P_2. Only recently a new nonconforming brick element of fourteen DOFs with quadratic and cubic convergence in the energy and L_2 norms is introduced by Meng, Sheen, Luo, and Kim [5], which has the same type of DOFs but has only cubic polynomials added to P_2. And then, the authors compared these 14-node elements numerically, see [4]. Numerical tests showed that at least for second-order elliptic problems Meng-Sheen-Luo-Kim and part of Smith-Kidger elements are convergent with optimal order or with lower order.

Convergence analysis for Meng-Sheen-Luo-Kim element was done in [5] and is fairly easy because it satisfies the patch test of Irons [3], which implies that a successful P_k-nonconforming element needs to satisfy that on each interface the jump of adjacent polynomials be orthogonal to P_{k-1} polynomials on the interface. Unfortunately, it was found in mathematics that the patch test is neither necessary nor sufficient, see [9] and the reference therein. As showed in this paper, the Smith-Kidge elements can only pass lower order patch test or can not pass it, but give optimal convergence order from our numerical results or lower convergence order. Thus, the convergence analysis for Smith-Kidger element is quite different and complex. For the convergence of the nonconforming element which can not pass the patch test, Stummel, Zhongci Shi etc. have done a lot of work, see [14, 8, 10, 11].

In this paper, we will give convergence analysis for Smith-Kidger element for the second-order elliptic problem. We show that although the patch test can not be satisfied, Type 1, 2 and 5 Smith-Kidger element can also get optimal convergence order, and Type 6 element loses one order of accuracy. Furthermore, we also present a new brick element with the same DOFs, which is also convergent with optimal order. Finally, if the value at the eight vertices and the integration value of six faces are taken as the DOFs, then we can show that Type 1, 2, 5, 6 elements and the proposed new element can get optimal convergence order, which implies that Type 6 element improves one order of accuracy.

The paper is organized as follows. In section 2, we will introduce Smith-Kidger element and give the basis functions firstly. In section 3, we define the interpolation operator and present convergence analysis for Type 1 Smith-Kidger element. In section 4, we will analyze the other elements and present the corresponding error estimates very briefly. In section 5, a new 14-node brick element is proposed. Finally, in section 6, we will conclude our results.

2 The quadratic nonconforming brick elements

Let $\widehat{\mathbf{K}} = [-1,1]^3$ and denote the vertices and face-centroids by $V_j, 1 \le j \le 8$, and $M_k, 1 \le k \le 6$, respectively. (see Fig. 1)

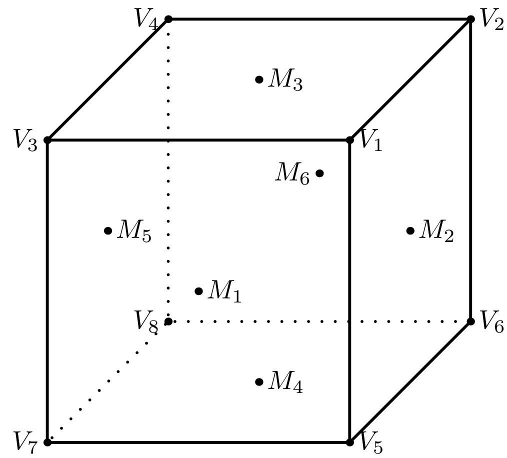

Figure 1: V_j denotes the vertices, $j = 1, 2, \ldots, 8$, and M_k denotes the face-centroid, $k = 1, 2, \ldots, 6$.

Smith and Kidger [12] defined the following six 14-node elements:

$$\begin{aligned}
\widehat{\mathbb{P}}_{SK}^{(1)} &= P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{xyz}, \widehat{x}^2\widehat{y}, \widehat{y}^2\widehat{z}, \widehat{z}^2\widehat{x}\}, && \text{(1a)}\\
\widehat{\mathbb{P}}_{SK}^{(2)} &= P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{xyz}, \widehat{xy}^2, \widehat{yz}^2, \widehat{zx}^2\}, && \text{(1b)}\\
\widehat{\mathbb{P}}_{SK}^{(3)} &= P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{xyz}, \widehat{x}^3, \widehat{y}^3, \widehat{z}^3\}, && \text{(1c)}\\
\widehat{\mathbb{P}}_{SK}^{(4)} &= P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{xyz}, \widehat{x}^2\widehat{yz}, \widehat{xy}^2\widehat{z}, \widehat{xyz}^2\}, && \text{(1d)}\\
\widehat{\mathbb{P}}_{SK}^{(5)} &= P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{xyz}, \widehat{x}^2\widehat{y} + \widehat{xy}^2, \widehat{y}^2\widehat{z} + \widehat{yz}^2, \widehat{z}^2\widehat{x} + \widehat{zx}^2\}, && \text{(1e)}\\
\widehat{\mathbb{P}}_{SK}^{(6)} &= P_2(\widehat{\mathbf{K}}) \oplus \mathrm{Span}\{\widehat{xyz}, \widehat{xy}^2\widehat{z}^2, \widehat{x}^2\widehat{yz}^2, \widehat{x}^2\widehat{y}^2\widehat{z}\}, && \text{(1f)}
\end{aligned}$$

whose DOFs are the function values at the eight vertices and the six face-centroids. They reported that Type 3 element fails and is inadmissible. We also remark that Type 4 element is also inadmissible since $(\widehat{x}^2 - 1)\widehat{yz} \in \widehat{\mathbb{P}}_{SK}^{(4)}$ vanishes at all these points. In [4], we observe that Type 1 (and 2) and Type 5 elements give optimal convergence results both in L^2 and H^1 norms at least for the second-order elliptic problems, while Type 6 element loses one order of accuracy in each norm.

In what follows, we will give the error estimate for the first type Smith-Kidger element detailedly and the error estimates for other types can be done completely similarly and are stated very briefly.

Firstly, the basis function of first type

$$\widehat{\phi}_1 = -\frac{1}{16} + \frac{1}{16}\left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right) + \frac{1}{8}\left(\widehat{xy} + \widehat{xz} + \widehat{yz} + \widehat{xyz} + \widehat{x}^2\widehat{y} + \widehat{y}^2\widehat{z} + \widehat{z}^2\widehat{x}\right),$$
$$\widehat{\phi}_2 = -\frac{1}{16} + \frac{1}{16}\left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right) + \frac{1}{8}\left(\widehat{xy} - \widehat{xz} - \widehat{yz} - \widehat{xyz} + \widehat{x}^2\widehat{y} - \widehat{y}^2\widehat{z} + \widehat{z}^2\widehat{x}\right),$$

$$
\begin{aligned}
\widehat{\phi}_3 &= -\frac{1}{16} + \frac{1}{16}\left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right) + \frac{1}{8}\left(-\widehat{x}\widehat{y} + \widehat{x}\widehat{z} - \widehat{y}\widehat{z} - \widehat{x}\widehat{y}\widehat{z} - \widehat{x}^2\widehat{y} + \widehat{y}^2\widehat{z} + \widehat{z}^2\widehat{x}\right), \\
\widehat{\phi}_4 &= -\frac{1}{16} + \frac{1}{16}\left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right) + \frac{1}{8}\left(-\widehat{x}\widehat{y} - \widehat{x}\widehat{z} - \widehat{y}\widehat{z} + \widehat{x}\widehat{y}\widehat{z} - \widehat{x}^2\widehat{y} - \widehat{y}^2\widehat{z} + \widehat{z}^2\widehat{x}\right), \\
\widehat{\phi}_5 &= -\frac{1}{16} + \frac{1}{16}\left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right) + \frac{1}{8}\left(-\widehat{x}\widehat{y} - \widehat{x}\widehat{z} + \widehat{y}\widehat{z} - \widehat{x}\widehat{y}\widehat{z} + \widehat{x}^2\widehat{y} + \widehat{y}^2\widehat{z} - \widehat{z}^2\widehat{x}\right), \\
\widehat{\phi}_6 &= -\frac{1}{16} + \frac{1}{16}\left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right) + \frac{1}{8}\left(-\widehat{x}\widehat{y} + \widehat{x}\widehat{z} - \widehat{y}\widehat{z} + \widehat{x}\widehat{y}\widehat{z} + \widehat{x}^2\widehat{y} - \widehat{y}^2\widehat{z} - \widehat{z}^2\widehat{x}\right), \\
\widehat{\phi}_7 &= -\frac{1}{16} + \frac{1}{16}\left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right) + \frac{1}{8}\left(\widehat{x}\widehat{y} - \widehat{x}\widehat{z} - \widehat{y}\widehat{z} + \widehat{x}\widehat{y}\widehat{z} - \widehat{x}^2\widehat{y} + \widehat{y}^2\widehat{z} - \widehat{z}^2\widehat{x}\right), \\
\widehat{\phi}_8 &= -\frac{1}{16} + \frac{1}{16}\left(\widehat{x}^2 + \widehat{y}^2 + \widehat{z}^2\right) + \frac{1}{8}\left(\widehat{x}\widehat{y} + \widehat{x}\widehat{z} + \widehat{y}\widehat{z} - \widehat{x}\widehat{y}\widehat{z} - \widehat{x}^2\widehat{y} - \widehat{y}^2\widehat{z} - \widehat{z}^2\widehat{x}\right), \\
\widehat{\phi}_9 &= \frac{1}{4} + \frac{1}{2}\widehat{x} + \frac{1}{4}\left(\widehat{x}^2 - \widehat{y}^2 - \widehat{z}^2\right) - \frac{1}{2}\widehat{z}^2\widehat{x}, \\
\widehat{\phi}_{10} &= \frac{1}{4} - \frac{1}{2}\widehat{x} + \frac{1}{4}\left(\widehat{x}^2 - \widehat{y}^2 - \widehat{z}^2\right) + \frac{1}{2}\widehat{z}^2\widehat{x}, \\
\widehat{\phi}_{11} &= \frac{1}{4} + \frac{1}{2}\widehat{y} + \frac{1}{4}\left(\widehat{y}^2 - \widehat{x}^2 - \widehat{z}^2\right) - \frac{1}{2}\widehat{x}^2\widehat{y}, \\
\widehat{\phi}_{12} &= \frac{1}{4} - \frac{1}{2}\widehat{y} + \frac{1}{4}\left(\widehat{y}^2 - \widehat{x}^2 - \widehat{z}^2\right) + \frac{1}{2}\widehat{x}^2\widehat{y}, \\
\widehat{\phi}_{13} &= \frac{1}{4} + \frac{1}{2}\widehat{z} + \frac{1}{4}\left(\widehat{z}^2 - \widehat{y}^2 - \widehat{x}^2\right) - \frac{1}{2}\widehat{y}^2\widehat{z}, \\
\widehat{\phi}_{14} &= \frac{1}{4} - \frac{1}{2}\widehat{z} + \frac{1}{4}\left(\widehat{z}^2 - \widehat{y}^2 - \widehat{x}^2\right) + \frac{1}{2}\widehat{y}^2\widehat{z}.
\end{aligned}
$$

Assume that $\Omega \in \mathbb{R}^3$ is a parallelepiped domain with boundary Γ. Let $(\mathscr{T}_h)_{h>0}$ be a regular family of triangulation of Ω into parallelepipeds $\mathbf{K}_j, j = 1, 2, \dots, N_{\mathbf{K}}$, where $h = \max_{\mathbf{K}\in\mathscr{T}_h} h_{\mathbf{K}}$ with $h_{\mathbf{K}} = \mathrm{diam}(\mathbf{K})$. For each $\mathbf{K} \in \mathscr{T}_h$, let $T_{\mathbf{K}} : \widehat{\mathbf{K}} \to \mathbb{R}^3$ be an invertible affine mapping such that

$$\mathbf{K} = T_{\mathbf{K}}(\widehat{\mathbf{K}}),$$

and denote $\phi_{\mathbf{K}} = \widehat{\phi} \circ T_{\mathbf{K}}^{-1} : \mathbf{K} \to \mathbb{R}$ for all $\widehat{\phi} \in \widehat{\mathbb{P}}_{SK}^{(1)}$, whose collection will be designated by

$$\mathbb{P}_{\mathbf{K}} = \mathrm{Span}\{\phi_{\mathbf{K}},\, \widehat{\phi} \in \widehat{\mathbb{P}}_{SK}^{(1)}\}.$$

Let N_V and N_F denote the numbers of vertices and faces, respectively. Then set

$$
\begin{aligned}
\mathscr{V}_h &= \{V_1, V_2, \dots, V_{N_V} : \quad \text{the set of all vertices of } \mathbf{K} \in \mathscr{T}_h\}, \\
\mathscr{F}_h &= \{F_1, F_2, \dots, F_{N_F} : \quad \text{the set of all faces of } \mathbf{K} \in \mathscr{T}_h\}, \\
\mathscr{M}_h &= \{M_1, M_2, \dots, M_{N_F} : \quad \text{the set of all face-centroids on } \mathscr{F}_h\}.
\end{aligned}
$$

We consider the following nonconforming finite element spaces:

$$
\begin{aligned}
\mathscr{N}\mathscr{C}^h &= \{\phi : \Omega \to \mathbb{R} |\ \phi|_{\mathbf{K}} \in \mathbb{P}_{\mathbf{K}} \forall \mathbf{K} \in \mathscr{T}_h, \phi \text{ is continuous at all } V_j \in \mathscr{V}_h, M_k \in \mathscr{M}_h\}, \\
\mathscr{N}\mathscr{C}_0^h &= \{\phi \in \mathscr{N}\mathscr{C}^h |\ \phi(V) = 0\ \forall V_j \in \mathscr{V}_h \cap \Gamma \text{ and } \phi(M) = 0\ \forall M_k \in \mathscr{M}_h \cap \Gamma\}.
\end{aligned}
$$

3 The interpolation operator and convergence analysis

In this section we will define an interpolation operator and analyze convergence in the case of Dirichlet problem. The case of Neumann problem is quite similar and the results will be omitted.

3.1 The second order elliptic problem

Denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product and (f,v) will be understood as the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, which is an extension of the duality paring between $L^2(\Omega)$. By $\|\cdot\|_k$ and $|\cdot|_k$ we adopt the standard notations for the norm and seminorm for the Sobolev spaces $H^k(\Omega)$. Consider then the following Dirichlet problem:

$$-\nabla\cdot(\boldsymbol{\alpha}\nabla u)+\beta u = f, \quad \Omega, \tag{2a}$$

$$u = 0, \quad \Gamma, \tag{2b}$$

with $\boldsymbol{\alpha}=(\alpha_{jk}), \alpha_{jk}, \beta \in L^\infty(\Omega), j,k=1,2,3, 0<\alpha_*|\boldsymbol{\xi}|^2 \le \boldsymbol{\xi}^t\boldsymbol{\alpha}(x)\boldsymbol{\xi} \le \alpha^*|\boldsymbol{\xi}|^2 < \infty, \boldsymbol{\xi}\in\mathbb{R}^3, \beta(x)\ge 0, x\in\Omega$, and $f\in H^1(\Omega)$. We will assume that the coefficients are sufficiently smooth and that the elliptic problem (2) has an $H^3(\Omega)$-regular solution such that $\|u\|_3 \le C\|f\|_1$. The weak problem is then given as usual: find $u\in H_0^1(\Omega)$ such that

$$a(u,v)=(f,v), \quad v\in H_0^1(\Omega), \tag{3}$$

where $a: H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{R}$ is the bilinear form defined by $a(u,v)=(\boldsymbol{\alpha}\nabla u,\nabla v)+(\beta u,v)$ for all $u,v\in H_0^1(\Omega)$. The nonconforming method for Problem (2) states as follows: find $u_h\in\mathscr{NC}_0^h$ such that

$$a_h(u_h,v_h)=(f,v_h), \quad v_h\in\mathscr{NC}_0^h, \tag{4}$$

where

$$a_h(u,v)=\sum_{\mathbf{K}\in\mathscr{T}_h} a_{\mathbf{K}}(u,v),$$

with $a_{\mathbf{K}}$ being the restriction of a to $\mathbf{K}$.

Notice that in order to have point values defined properly we need to recall Sobolev embedding theorem

$$W^{m,p}(\Omega)\longrightarrow C^0(\Omega), \text{ if } \frac{1}{p}-\frac{m}{d}<0.$$

Thus we should have $p>\frac{m}{3}$. For a given cube $\mathbf{K}\in\mathscr{T}_h$, define the local interpolation operator $\Pi_{\mathbf{K}}: W^{1,p}(\mathbf{K})\cap H_0^1(\Omega)\longrightarrow\widehat{\mathbb{P}}_{SK}^{(1)},\ p>3$, by

$$\Pi_{\mathbf{K}}\phi(V_i)=\phi(V_i), \quad \Pi_{\mathbf{K}}\phi(M_j)=\phi(M_j)$$

for all vertices V_i and face-centroids M_j of $\mathbf{K}$. The global interpolation operator Π_h: $W^{1,p}(\Omega)\cap H_0^1(\Omega)\to\mathscr{NC}_0^h$ is then defined through the local interpolation operator $\Pi_{\mathbf{K}}$

by $\Pi_h|_{\mathbf{K}} = \Pi_{\mathbf{K}}$ for all $\mathbf{K} \in \mathscr{T}_h$. Since Π_h preserves P_2 for all $\mathbf{K} \in \mathscr{T}_h$, it follows from the Bramble-Hilbert Lemma that

$$\sum_{\mathbf{K}\in\mathscr{T}_h} \|\phi - \Pi_h\phi\|_{L^2(\mathbf{K})} + h \sum_{\mathbf{K}\in\mathscr{T}_h} \|\phi - \Pi_h\phi\|_{H^1(\mathbf{K})} \leq Ch^k \|\phi\|_{H^k(\Omega)}, \tag{5}$$
$$\phi \in W^{k,p}(\Omega) \cap H_0^1(\Omega), 1 \leq k \leq 3.$$

We now consider the energy-norm error estimate and first consider the following Strang lemma [13].

Lemma 1. *Let $u \in H^1(\Omega)$ and $u_h \in \mathscr{N}\mathscr{C}_0^h$ be the solutions of Eq. (3) and Eq. (4), respectively. Then*

$$\|u - u_h\|_h \leq c\Big\{ \inf_{v\in\mathscr{N}\mathscr{C}_0^h} \|u - v\|_h + \sup_{w\in\mathscr{N}\mathscr{C}_0^h} \frac{|a_h(u,w) - \langle f, w\rangle|}{\|w\|_h} \Big\}. \tag{6}$$

Assume sufficient regularity such that $u \in H^3$. Due to (5), the first term in the right side of (6) is bounded by

$$\inf_{v\in\mathscr{N}\mathscr{C}_0^h} \|u - v\|_h \leq \|u - \Pi_h u\|_h \leq ch^s |u|_{H^{s+1}(\Omega)}, 1 < s \leq 2 \tag{7}$$

Now let us bound the second term of the right side of (6) which denotes the consistency error. For a given cube $\mathbf{K} \in \mathscr{T}_h$, denote by F_K^{x+} and F_K^{x-} the face of $\mathbf{K}$ with outward normal $(1, 0, 0)$ and $(-1, 0, 0)$, respectively. By the same way, we denote the other faces by $F_K^{y+}, F_K^{y-}, F_K^{z+}, F_K^{z-}$. Thus, integrating by parts elementwise, we have

$$\begin{aligned}
a_h(u,w) - \langle f, w\rangle &= \sum_{\mathbf{K}\in\mathscr{T}_h} \langle \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u, w\rangle_{\partial\mathbf{K}} \\
&= \sum_{\mathbf{K}\in\mathscr{T}_h} \int_{F_K^{x+}\cup F_K^{x-}} \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u w \mathrm{d}s + \sum_{\mathbf{K}\in\mathscr{T}_h} \int_{F_K^{y+}\cup F_K^{y-}} \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u w \mathrm{d}s \\
&\quad + \sum_{\mathbf{K}\in\mathscr{T}_h} \int_{F_K^{z+}\cup F_K^{z-}} \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u w \mathrm{d}s \\
&:= E_x + E_y + E_z
\end{aligned}$$

where $\boldsymbol{\nu}^T$ is the transpose of the unit outward normal to $\mathbf{K}$.

Before proceeding, we need the following lemma.

Lemma 2. *By F_k denote the face containing the centroid M_k and by $V_j^{F_k}, j = 1, 2, 3, 4$, denote the vertices on the surface F_k. If $p \in \widehat{\mathbb{P}}_{SK}^{(1)}$, $p(V_j^{F_k}) = 0, j = 1, 2, 3, 4$, and $p(M_k) = 0$, then*

$$\int_{F_k} p(x,y,z)\mathrm{d}s = 0, \quad k = 1, 2, \ldots, 6. \tag{8}$$

Proof. Without loss of generality, we assume that $M_1 = (1, 0, 0)$. In this case, we have $p \in \widehat{\mathbb{P}}^{(1)}_{SK}$ and $p(1, \pm 1, \pm 1) = p(1, 0, 0) = 0$. We need to prove that

$$\int_{F_1} p(1, y, z)\, \mathrm{d}y\mathrm{d}z = 0. \tag{9}$$

It follows from $p(1, \pm 1, \pm 1) = 0$ that

$$p(1, y, z) = l_1(y, z)(y^2 - 1) + l_2(y, z)(z^2 - 1), \quad l_j \in P_1(\mathbb{R}^2),\ j = 1, 2. \tag{10}$$

Set

$$l_j(y, z) = a_j y + b_j z + c_j, \quad j = 1, 2.$$

Then $p(1, 0, 0) = 0$ implies that $c_1 + c_2 = 0$, which reduces (10) to

$$p(1, y, z) = (a_1 y + b_1 z)(y^2 - 1) + (a_2 y + b_2 z)(z^2 - 1) + c_1(y^2 - z^2).$$

Since

$$\widehat{\mathbb{P}}^{(1)}_{SK}|_{x=1} = \mathrm{Span}\{1, y, z, y^2, yz, z^2, y^2 z\}, \tag{11}$$

and notice that $p \in \widehat{\mathbb{P}}^{(1)}_{SK}$, we have

$$a_1 = a_2 = b_2 = 0,$$

which leads to

$$p(1, y, z) = b_1 z(y^2 - 1) + c_1(y^2 - z^2). \tag{12}$$

By a direct computation using (12), Eq. (8) holds. This completes the proof. $\square$

This lemma implies that Type 1 element can pass lower order patch test (test functions are in P_0 not P_1), which will lead to a convergence solution for the second order elliptic problems, but the convergence order is not optimal. To bound E_x, E_y, E_z, we also need some interpolation operators.

3.2 Some interpolation operators

For the reference element $\widehat{\mathbf{K}} = [-1, 1] \times [-1, 1] \times [-1, 1]$, we consider the interpolation problem on the face of $F^{x+}_{\widehat{\mathbf{K}}}$. The interpolation points are $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, 1)$, $(1, -1, -1)$, $(1, 0, 0)$ which are the four vertices and the centroid of $F^{x+}_{\widehat{\mathbf{K}}}$, and the interpolation space is $\mathrm{Span}\{1, \widehat{y}, \widehat{z}, \widehat{yz}, \widehat{z}^2\}$. This interpolation problem is poised and the basis functions are

$$\begin{aligned}
\widehat{\varphi}_1 &= \frac{1}{4}\widehat{y} + \frac{1}{4}\widehat{z} + \frac{1}{4}\widehat{yz} + \frac{1}{4}\widehat{z}^2, \\
\widehat{\varphi}_2 &= \frac{1}{4}\widehat{y} - \frac{1}{4}\widehat{z} - \frac{1}{4}\widehat{yz} + \frac{1}{4}\widehat{z}^2, \\
\widehat{\varphi}_3 &= -\frac{1}{4}\widehat{y} + \frac{1}{4}\widehat{z} - \frac{1}{4}\widehat{yz} + \frac{1}{4}\widehat{z}^2,
\end{aligned}$$

$$
\begin{aligned}
\widehat{\varphi}_4 &= -\frac{1}{4}\widehat{y} - \frac{1}{4}\widehat{z} + \frac{1}{4}\widehat{y}\widehat{z} + \frac{1}{4}\widehat{z}^2, \\
\widehat{\varphi}_5 &= 1 - \widehat{z}^2.
\end{aligned}
$$

Thus for a function f defined on the face $F_{\widehat{\mathbf{K}}}^{x+}$, the interpolation polynomial can be written as

$$
\widehat{T}_F^{x+} f = f(1,1,1)\widehat{\varphi}_1 + f(1,1,-1)\widehat{\varphi}_2 + f(1,-1,1)\widehat{\varphi}_3 + f(1,-1,-1)\widehat{\varphi}_4 + f(1,0,0)\widehat{\varphi}_5
$$

For a given $\mathbf{K} \in \mathscr{T}_h$, we can define the interpolation operator T_F^{x+} by $T_F^{x+} = \widehat{T}_F^{x+} \circ T_{\mathbf{K}}^{-1}$. Similarly, we can also define the interpolation operators on the other five faces of $\mathbf{K}$ and denote them by $T_F^{x-}, T_F^{y+}, T_F^{y-}, T_F^{z+}, T_F^{z-}$, respectively. Here we remark that the interpolation space of T_F^{x-} is same with T_F^{x+}, the interpolation space of T_F^{y+} and T_F^{y-} is $\mathrm{Span}\{1, x, z, xz, x^2\}$, and the interpolation space of T_F^{z+} and T_F^{z-} is $\mathrm{Span}\{1, x, y, xy, y^2\}$.

Define $RQ = \mathrm{Span}\{1, \widehat{x}, \widehat{y}, \widehat{z}, \widehat{x}^2 - \widehat{y}^2, \widehat{x}^2 - \widehat{z}^2\}$. For the reference element $\widehat{\mathbf{K}}$, let R_K be a interpolation operator $R_{\widehat{\mathbf{K}}} : H^2(\widehat{\mathbf{K}}) \to RQ$ defined by

$$
R_{\widehat{\mathbf{K}}}\widehat{\phi}(\widehat{M}_j) = \widehat{\phi}(\widehat{M}_j), j = 1, \ldots, 6
$$

for all $\widehat{\phi} \in H^2(\widehat{\mathbf{K}})$. It is exactly the so-called rotation element. Obviously,

$$
R_{\widehat{\mathbf{K}}}\widehat{\phi} = \sum_{i=1}^{6} \widehat{\phi}(\widehat{M}_i)\widehat{\psi}_i(\widehat{x}, \widehat{y}, \widehat{z})
$$

where

$$
\begin{aligned}
\widehat{\psi}_1 &= -\frac{1}{6}\widehat{y}^2 - \frac{1}{6}\widehat{z}^2 + \frac{1}{3}\widehat{x}^2 + \frac{1}{2}\widehat{x} + \frac{1}{6}, \\
\widehat{\psi}_2 &= -\frac{1}{6}\widehat{y}^2 - \frac{1}{6}\widehat{z}^2 + \frac{1}{3}\widehat{x}^2 - \frac{1}{2}\widehat{x} + \frac{1}{6}, \\
\widehat{\psi}_3 &= -\frac{1}{6}\widehat{x}^2 - \frac{1}{6}\widehat{z}^2 + \frac{1}{3}\widehat{y}^2 + \frac{1}{2}\widehat{y} + \frac{1}{6}, \\
\widehat{\psi}_4 &= -\frac{1}{6}\widehat{x}^2 - \frac{1}{6}\widehat{z}^2 + \frac{1}{3}\widehat{y}^2 - \frac{1}{2}\widehat{y} + \frac{1}{6}, \\
\widehat{\psi}_5 &= -\frac{1}{6}\widehat{x}^2 - \frac{1}{6}\widehat{y}^2 + \frac{1}{3}\widehat{z}^2 + \frac{1}{2}\widehat{z} + \frac{1}{6}, \\
\widehat{\psi}_6 &= -\frac{1}{6}\widehat{x}^2 - \frac{1}{6}\widehat{y}^2 + \frac{1}{3}\widehat{z}^2 + \frac{1}{2}\widehat{z} + \frac{1}{6}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
R_{\widehat{\mathbf{K}}}\widehat{\phi}|_{\widehat{x}=1} &= \sum_{i=1}^{6} \widehat{\phi}(\widehat{M}_i)\widehat{\psi}_i(1, \widehat{y}, \widehat{z}) \\
&= \sum_{i=3}^{6} \widehat{\phi}(\widehat{M}_i)\widehat{\psi}_i(1, \widehat{y}, \widehat{z}) + \widehat{\phi}(\widehat{M}_1)\left(-\frac{1}{6}\widehat{y}^2 - \frac{1}{6}\widehat{z}^2 + 1\right) \\
&\quad + \widehat{\phi}(\widehat{M}_2)\left(-\frac{1}{6}\widehat{y}^2 - \frac{1}{6}\widehat{z}^2\right)
\end{aligned}
$$

$$= \Theta(\widehat{\mathbf{K}}, \widehat{\phi}, \widehat{y}, \widehat{z}) + \widehat{\phi}(\widehat{M}_1).$$

It is easy to notice that $\widehat{\psi}_i(1, \widehat{y}, \widehat{z}) = \widehat{\psi}_i(-1, \widehat{y}, \widehat{z})$ for $i = 3, 4, 5, 6$, which will leads to

$$\begin{aligned}
R_{\widehat{\mathbf{K}}}\widehat{\phi}|_{\widehat{x}=-1} &= \sum_{i=1}^{6} \widehat{\phi}(\widehat{M}_i)\widehat{\psi}_i(1, \widehat{y}, \widehat{z}) \\
&= \sum_{i=3}^{6} \widehat{\phi}(\widehat{M}_i)\widehat{\psi}_i(1, \widehat{y}, \widehat{z}) + \widehat{\phi}(\widehat{M}_1)\left(-\frac{1}{6}\widehat{y}^2 - \frac{1}{6}\widehat{z}^2\right) \\
&\quad + \widehat{\phi}(\widehat{M}_2)\left(-\frac{1}{6}\widehat{y}^2 - \frac{1}{6}\widehat{z}^2 + 1\right) \\
&= \Theta(\widehat{\mathbf{K}}, \widehat{\phi}, \widehat{y}, \widehat{z}) + \widehat{\phi}(\widehat{M}_2).
\end{aligned}$$

For any given $\mathbf{K} \in \mathscr{T}$, we can define the interpolation operator $R_{\mathbf{K}} := R_{\widehat{\mathbf{K}}} \cdot T_K^{-1}$. Denote by M_K^{x+} and M_K^{x-} the centroids of the faces with outward normal $(1, 0, 0)$ and $(-1, 0, 0)$, respectively. Then for any $\phi \in H^2(\mathbf{K})$, we have

$$\begin{aligned}
R_{\mathbf{K}}\phi|_{x=1} &= \Theta(\mathbf{K}, \phi, y, z) + \phi(M_K^{x+}), \\
R_{\mathbf{K}}\phi|_{x=-1} &= \Theta(\mathbf{K}, \phi, y, z) + \phi(M_K^{x-}).
\end{aligned}$$

3.3 The error estimates

Let us give an estimation of E_x and the estimation of E_y and E_z can be done similarly. According to the definition of E_x and T_F^{x+}, T_F^{x-}, we have

$$\begin{aligned}
E_x &= \sum_{\mathbf{K}\in\mathscr{T}_h} \int_{F_K^{x+}\cup F_K^{x-}} \boldsymbol{\nu}_j^T \boldsymbol{\alpha} \nabla u_j w \mathrm{d}s \\
&= \sum_{\mathbf{K}\in\mathscr{T}_h} \left(\int_{F_K^{x+}} \boldsymbol{\nu}_j^T \boldsymbol{\alpha} \nabla u_j (w - T_F^{x+}(w)) \mathrm{d}s + \int_{F_K^{x-}} \boldsymbol{\nu}_j^T \boldsymbol{\alpha} \nabla u_j (w - T_F^{x-}(w)) \mathrm{d}s \right).
\end{aligned}$$

Next we want to prove

$$(w|_{F_K^{x+}} - T_F^{x+}(w)) = (w|_{F_K^{x-}} - T_F^{x-}(w)), \tag{13}$$

which can be done on the reference element. That is, we need to show

$$(\widehat{w}|_{\widehat{F}_{\widehat{\mathbf{K}}}^{\widehat{x}+}} - \widehat{T}_{\widehat{F}}^{\widehat{x}+}(\widehat{w})) = (\widehat{w}|_{\widehat{F}_{\widehat{\mathbf{K}}}^{\widehat{x}-}} - \widehat{T}_{\widehat{F}}^{\widehat{x}-}(\widehat{w})). \tag{14}$$

Due to $\widehat{w} \in \widehat{\mathbb{P}}_{SK}^{(1)}$, we have

$$\widehat{w} = \sum_{i=1}^{14} \alpha_i \widehat{\phi}_i.$$

Direct computation will derive

$$(\widehat{\phi}_i|_{\widehat{F}_{\widehat{\mathbf{K}}}^{\widehat{x}+}} - \widehat{T}_{\widehat{F}}^{\widehat{x}+}(\widehat{\phi}_i)) = (\widehat{\phi}_i|_{\widehat{F}_{\widehat{\mathbf{K}}}^{\widehat{x}-}} - \widehat{T}_{\widehat{F}}^{\widehat{x}-}(\widehat{\phi}_i)), \tag{15}$$

which will ensure Eq. (14). Denote by $M_F(u)$ the value of $\boldsymbol{\nu}_j^T \boldsymbol{\alpha} \nabla u$ at the centroid of F. Thus we arrive at

$$
\begin{aligned}
E_x &= \sum_{\mathbf{K}\in\mathscr{T}_h} \left(\int_{F_K^{x+}} \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u (w - T_F^{x+}(w)) \mathrm{d}s + \int_{F_K^{x-}} \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u (w - T_F^{x-}(w)) \mathrm{d}s \right) \\
&= \sum_{\mathbf{K}\in\mathscr{T}_h} \left(\int_{F_K^{x+}} \left(\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u - M_{F_K^{x+}}(u) \right) (w - T_F^{x+}(w)) \mathrm{d}s \right. \\
&\quad \left. + \int_{F_K^{x-}} \left(\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u - M_{F_K^{x-}}(u) \right) (w - T_F^{x-}(w)) \mathrm{d}s \right) \\
&= \sum_{\mathbf{K}\in\mathscr{T}_h} \left(\int_{F_K^{x+}} \left(\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u - \Theta(\mathbf{K}, u, y, z) - M_{F_K^{x+}}(u) \right) (w - T_F^{x+}(w)) \mathrm{d}s \right. \\
&\quad \left. + \int_{F_K^{x-}} \left(\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u - \Theta(\mathbf{K}, u, y, z) - M_{F_K^{x-}}(u) \right) (w - T_F^{x-}(w)) \mathrm{d}s \right) \\
&= \sum_{\mathbf{K}\in\mathscr{T}_h} \left(\int_{F_K^{x+}} \left(\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u - R_{\mathbf{K}} u \right) (w - T_F^{x+}(w)) \mathrm{d}s \right. \\
&\quad \left. + \int_{F_K^{x-}} \left(\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u - R_{\mathbf{K}} u \right) (w - T_F^{x-}(w)) \mathrm{d}s \right)
\end{aligned}
$$

where $R_{\mathbf{K}} u$ is an interpolation polynomial of $\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u$. Here we remark that in the second step $-M_F(u)$ is added due to the orthogonality (8) and in the third step $-\Theta(\mathbf{K}, u, y, z)$ is added due to Eq. (14). We also mention that $R_{\mathbf{K}} u$ can not be added directly according to Eq. (14) because $R_{\mathbf{K}} u$ takes different value on the face F_K^{x+} and F_K^{x-}. Since $R_{\mathbf{K}}$ and T_F^x preserves $P_1(\mathbf{K})$ and $P_1(F_K^x)$, respectively, it follows from trace theorem and Cauchy-Schwartz inequality, we get

$$|E_x| \leq Ch^2 ||w||_h ||u||_{H^3(\Omega)}.$$

Similarly, we also have

$$|E_y| \leq Ch^2 ||w||_h ||u||_{H^3(\Omega)}, \ |E_z| \leq Ch^2 ||w||_h ||u||_{H^3(\Omega)}.$$

Hence

$$\sup_{w\in\mathscr{N}\mathscr{C}_0^h} \frac{|a_h(u,w) - \langle f, w\rangle|}{\|w\|_h} = \sup_{w\in\mathscr{N}\mathscr{C}_0^h} \frac{|E_x + E_y + E_z|}{||w||_h} \leq Ch^2 ||u||_{H^3(\Omega)}.$$

By collecting the above results, we get the following energy-norm error estimate.

Theorem 1. *Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ and $u_h \in \mathscr{N}\mathscr{C}_0^h$ satisfy (3) and (4), respectively. Then we have the energy norm error estimate:*

$$||u - u_h||_h \leq Ch^2 ||u||_{H^3(\Omega)}.$$

By the standard Aubin-Nitsche duality argument, the $L_2(\Omega)$-error estimate can be easily obtained, but the corresponding proof is omitted. We state the result in the following theorem.

Theorem 2. *Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ and $u_h \in \mathscr{N}\mathscr{C}_0^h$ be the solution of (3) and (4), respectively. Then we have*

$$\|u - u_h\|_0 \leq Ch^3 \|u\|_{H^3(\Omega)}.$$

4 Error estimates of other Smith-Kidger element

In this section we claim that solving second-order elliptic problem with Smith-Kidger element of type 2 and 5 is also convergent with optimal order. In these case, it is easy to check that the orthogonality in Lemma 2 holds. The difference during the proof lies in the construction of the interpolation operator. For the second type element, the interpolation of T_k^x, T_k^y, T_k^z should be Span$\{1, y, z, yz, y^2\}$, Span$\{1, x, z, xz, z^2\}$ and Span$\{1, x, y, xy, x^2\}$, respectively. And for the fifth type element, the corresponding interpolation spaces should be taken as Span$\{1, y, z, yz, z^2 + y^2\}$, Span$\{1, x, z, xz, x^2 + z^2\}$ and Span$\{1, x, y, xy, x^2 + y^2\}$, respectively.

For Type 6 element, the orthogonality in Lemma 2 does not hold, but the Eq. (14) hods. Thus, we have

$$\begin{aligned}
|E_x| &= \left| \sum_{\mathbf{K} \in \mathscr{T}_h} \left(\int_{F_K^{x+}} \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u (w - T_F^{x+}(w)) \mathrm{d}s + \int_{F_K^{x-}} \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u (w - T_F^{x-}(w)) \mathrm{d}s \right) \right| \\
&= \left| \sum_{\mathbf{K} \in \mathscr{T}_h} \left(\int_{F_K^{x+}} \left(\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u - P_{\mathbf{K}}^0 (\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u) \right) (w - T_F^{x+}(w)) \mathrm{d}s \right.\right. \\
&\quad \left.\left. + \int_{F_K^{x-}} \left(\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u - P_{\mathbf{K}}^0 (\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u) \right) (w - T_F^{x-}(w)) \mathrm{d}s \right) \right| \\
&\leq Ch ||u||_2 ||w||_h
\end{aligned}$$

where

$$P_{\mathbf{K}}^0 (\boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u) = \frac{1}{|\mathbf{K}|} \int_{\mathbf{K}} \boldsymbol{\nu}^T \boldsymbol{\alpha} \nabla u \mathrm{d}s$$

and $|\mathbf{K}| = \int_{\mathbf{K}} \mathrm{d}s$. By a similar derivation, we will get

Theorem 3. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy (3) and u_h be the solution of (4) with the sixth type element. Then we have the energy norm error estimate:*

$$\begin{aligned}
||u - u_h||_h &\leq Ch ||u||_{H^2(\Omega)}, \\
||u - u_h||_0 &\leq Ch^2 \|u\|_{H^2(\Omega)}.
\end{aligned}$$

5 A new 14-node brick element

In this section, we present a new element with 14-node. The degrees of freedom are the same with those in Smith-Kidger element and Meng-Sheen-Luo-Kim element. But the shape function space is taken as $P_2 \oplus \mathrm{Span}\{x(y^2+z^2), y(x^2+z^2), z(x^2+y^2)\}$. The space can match the DOFs and the basis functions are

$$\begin{aligned}
\widehat{\theta}_1 &= \frac{1}{16}\left(-1+x^2+y^2+z^2+x(y^2+z^2)+y(x^2+z^2)+z(x^2+y^2)\right)\\
&\qquad +\frac{1}{8}\left(xy+xz+yz+xyz\right),\\
\widehat{\theta}_2 &= \frac{1}{16}\left(-1+x^2+y^2+z^2+x(y^2+z^2)+y(x^2+z^2)-z(x^2+y^2)\right)\\
&\qquad +\frac{1}{8}\left(xy-xz-yz-xyz\right),\\
\widehat{\theta}_3 &= \frac{1}{16}\left(-1+x^2+y^2+z^2+x(y^2+z^2)-y(x^2+z^2)+z(x^2+y^2)\right)\\
&\qquad +\frac{1}{8}\left(-xy+xz-yz-xyz\right),\\
\widehat{\theta}_4 &= \frac{1}{16}\left(-1+x^2+y^2+z^2+x(y^2+z^2)-y(x^2+z^2)-z(x^2+y^2)\right)\\
&\qquad +\frac{1}{8}\left(-xy-xz+yz+xyz\right),\\
\widehat{\theta}_5 &= \frac{1}{16}\left(-1+x^2+y^2+z^2-x(y^2+z^2)+y(x^2+z^2)+z(x^2+y^2)\right)\\
&\qquad +\frac{1}{8}\left(-xy-xz+yz-xyz\right),\\
\widehat{\theta}_6 &= \frac{1}{16}\left(-1+x^2+y^2+z^2-x(y^2+z^2)+y(x^2+z^2)-z(x^2+y^2)\right)\\
&\qquad +\frac{1}{8}\left(-xy+xz-yz+xyz\right),\\
\widehat{\theta}_7 &= \frac{1}{16}\left(-1+x^2+y^2+z^2-x(y^2+z^2)-y(x^2+z^2)+z(x^2+y^2)\right)\\
&\qquad +\frac{1}{8}\left(xy-xz-yz+xyz\right),\\
\widehat{\theta}_8 &= \frac{1}{16}\left(-1+x^2+y^2+z^2-x(y^2+z^2)-y(x^2+z^2)-z(x^2+y^2)\right)\\
&\qquad +\frac{1}{8}\left(xy+xz+yz-xyz\right),\\
\widehat{\theta}_9 &= \frac{1}{4}+\frac{1}{2}x+\frac{1}{4}\left(x^2-y^2-z^2\right)-\frac{1}{4}x(y^2+z^2),\\
\widehat{\theta}_{10} &= \frac{1}{4}-\frac{1}{2}x+\frac{1}{4}\left(x^2-y^2-z^2\right)+\frac{1}{4}x(y^2+z^2),\\
\widehat{\theta}_{11} &= \frac{1}{4}+\frac{1}{2}y+\frac{1}{4}\left(-x^2+y^2-z^2\right)-\frac{1}{4}y(x^2+z^2),\\
\widehat{\theta}_{12} &= \frac{1}{4}-\frac{1}{2}y+\frac{1}{4}\left(-x^2+y^2-z^2\right)+\frac{1}{4}y(x^2+z^2),
\end{aligned}$$

$$\widehat{\theta}_{13} = \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}\left(-x^2 - y^2 + z^2\right) - \frac{1}{4}z(x^2 + y^2),$$
$$\widehat{\theta}_{14} = \frac{1}{4} - \frac{1}{2}z + \frac{1}{4}\left(-x^2 - y^2 + z^2\right) + \frac{1}{4}z(x^2 + y^2).$$

To analyze the convergence, we need to verify the orthogonality in Lemma 2 and Eq. (14). The orthogonality can be checked directly like the proof in Lemma 2. To satisfy Eq. (14), we take the corresponding interpolation spaces as $\mathrm{Span}\{1, y, z, yz, z^2 + y^2\}$, $\mathrm{Span}\{1, x, z, xz, x^2 + z^2\}$ and $\mathrm{Span}\{1, x, y, xy, x^2 + y^2\}$, respectively. Thus, by the idea above, we can also get optimal convergence order for the second-order elliptic problems. That is, in this case, theorems 1 and 2 hold.

6 Further remarks and conclusions

In this paper, we have proved that for second-order elliptic problems, the Smith-Kidger element of type 1, 2 and 5 can obtain optimal convergence order both in energy norm and $L_2(\Omega)$ norm, while the sixth type element loses one order of accuracy in each norm. In the proof, the key points lie in that they have weak orthogonality (Lemma 2) and satisfy Eq. (14). In [5], we also proposed another kind of DOFs, that is, the value at the eight vertices and the integration value of six faces. Indeed, it is easy to check that Type 1, 2, 5 and the new element presented in this paper can get optimal convergence order for second-order elliptic problems. Besides, we can show that Type 6 also has the optimal convergence order because the weak orthogonality naturally hods, which improve one order accuracy in this case.

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